THE STRUCTURE OF QF-3 RINGS(1)

BY
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Introduction. In [12] Thrall introduced three generalizations of the quasi-Frobenius (=QF) algebras of Nakayama [9], [10]. In this paper we shall be concerned with ring theoretic generalizations of two of these Thrall algebras—namely QF-2 algebras and QF-3 algebras.

If R is a ring then a one-sided ideal of R is *primitive* in case it is generated by a primitive idempotent, and an R-module is *minimal faithful* in case it is faithful and has no proper faithful direct summand. Extending Thrall's original definitions to (two-sided) artinian rings we have:

QF-2 *rings*: An artinian ring is QF-2 in case each of its primitive one-sided ideals has a simple socle.

QF-3 *rings*: An artinian ring is QF-3 in case it has (to within isomorphism) a unique minimal faithful left module.

It is not difficult to show that QF rings are both QF-2 and QF-3 (see [2, §§58–59]). Moreover, Thrall [12] has shown that QF-2 algebras are QF-3 but not necessarily QF. Most of the information about QF-2 and QF-3 rings is limited to finite dimensional algebras (see [8], [12], [13]). Two notable exceptions generalize to QF-3 rings results known to hold for QF-3 algebras almost from their inception. Specifically, Jans [7] has characterized QF-3 rings as those artinian rings whose left injective hulls are projective and Harada [5] has shown that the QF-3 property is actually "two-sided" (i.e., a QF-3 ring has a unique minimal faithful right module).

In §2 of this paper we obtain ideal theoretic characterizations of the injective projective modules (and hence of the unique minimal faithful module) over "left QF-3 rings".

Our main results appear in §3. With the aid of Morita's duality theorems [8] we obtain characterizations of QF-3 rings that are analogous to Nakayama's original definition of QF rings in terms of socles of primitive one-sided ideals [10], his characterization of QF-rings in terms of the double annihilator property for one-sided ideals [10], and the fact (see [8, §14]) that QF rings are precisely those artinian rings for which the functor $\mathrm{Hom}_R(\ \ , R)$ provides a duality between the categories of finitely generated left and finitely generated right R-modules.

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Finally, in §4 we extend to artinian rings Thrall's theorem [12] that QF-2 algebras are QF-3.

1. **Preliminaries.** Throughout this paper R denotes an associative ring with identity 1 and Jacobson radical N. All R-modules are unitary. Unless otherwise specified we shall assume that R is left artinian. In such a ring R there is always an orthogonal set $e_1, \ldots, e_n$ of primitive idempotents such that $Re_1, \ldots, Re_n$ is a complete collection of pairwise nonisomorphic indecomposable projective left R-modules (see [4, p. 331]). We call such a set $e_1, \ldots, e_n$ a *basic* set of primitive idempotents for R.

If M is a left (right) R-module we write $T(M)=M/NM$ $(T(M)=M/MN)$, $S(M)$ for the socle of M and $E(M)=E(S(M))$ for the injective hull (see [3]) of M. If M has a composition series, $c(M)$ denotes its length.

We shall need to consider two types of annihilators. If M is a left (right) R-module and $T \subseteq R$

$$\mathrm{Ann}_R(M) = \{r \in R : rM = 0, (Mr = 0)\}$$

and

$$\mathrm{Ann}_M(T) = \{m \in M : Tm = 0\,(mT = 0)\}.$$

The first is a two-sided ideal, and the second is a submodule if T is a right (left) ideal. We abbreviate

$$l(T) = \mathrm{Ann}_{R_R}(T) \quad \text{and} \quad r(T) = \mathrm{Ann}_{_RR}(T).$$

These are always left and right ideals respectively. In particular, $S(_RR)=r(N)$ and $S(R_R)=l(N)$ are two-sided ideals.

Let us examine the result of Jans referred to in the introduction. He proved, on one hand, if R is semiprimary and $E(_RR)$ is projective, then R has a unique minimal faithful module that is injective, projective and isomorphic to a direct summand of every faithful left R-module [7, Theorem 3.2 and its proof]. On the other hand he proved that if R is right Noetherian and has a faithful injective left module that is imbeddable in every faithful left R-module, then $E(_RR)$ is projective [7, Theorem 3.1]. In particular then, Jans proved that a right artinian ring has a unique minimal faithful left module if and only if its injective hull as a left module is projective.

We can easily show that the latter result is true for left artinian rings by extending Thrall's proof for the case in which R is an algebra [12, Theorem 5]. In the process we have a good look at the unique minimal faithful module.

(1.1) Propositon. *Let R be a left artinian ring with basic set of primitive idempotents $e_1, \ldots, e_n$ and unique minimal faithful left module U. Then*

(a) *$e_1, \ldots, e_n$ can be numbered in such a manner that $U \cong Re_1 + \cdots + Re_m$ for some $m \leqq n$.*

(b) *$Re_1, \ldots, Re_m$ are injective.*

(c) *Every minimal left ideal is isomorphic to the socle of some unique* Re_k, $1 \leqq k \leqq m$.

(d) $E({}_RR)$ *is projective.*

(e) $Re_1, \ldots, Re_m$ *is a complete set of pairwise nonisomorphic indecomposable injective projective left R-modules.*

Proof. Let R, $e_1, \ldots, e_n$ and U be as in the hypothesis.

(a) The left module $Re_1 + \cdots + Re_n$ is faithful. Renumbering, we may assume $e_1, \ldots, e_m$ is a subset of $e_1, \ldots, e_n$ chosen minimal with respect to "$Re_1 + \cdots + Re_m$ is faithful". But then $Re_1 + \cdots + Re_m$ has no proper faithful direct summand, so $U \cong Re_1 + \cdots + Re_m$.

(b) Let $S_1, \ldots, S_t$ be a complete set of pairwise nonisomorphic minimal left ideals of R and let $E = E(S_1 \oplus \cdots \oplus S_t)$. Then E, being an injective module that contains a copy of each minimal left ideal of R, is faithful. Now note that $S_1 \oplus \cdots \oplus S_t = S(E)$, since it is an essential semisimple submodule of E, and that if $E = H \oplus K$ then $S(E) = S(H) \oplus S(K)$. Applying the Krull-Schmidt theorem to $S(E)$ it follows that a proper direct summand cannot contain copies of every minimal left ideal of R and hence cannot be faithful. Therefore $E \cong U$, and the indecomposable direct summands of E must be $Re_1, \ldots, Re_m$. This proves (b).

(c), (d) and (e) now follow from well-known properties of artinian rings and the injective hull.

Keeping Jans' result [7, Theorem 3.2] and (1.1) in mind we shall say that a left artinian ring R is *left* QF-3 in case R satisfies any of the following equivalent conditions.

(a) R has (to within isomorphism) a unique minimal faithful left module.

(b) $E({}_RR)$ is projective.

(c) R has a faithful injective projective left module. Note that, by the result of Harada [5] quoted in the introduction, a (two-sided) artinian ring is QF-3 if and only if it satisfies any one of (a), (b), (c) and their right-hand versions.

2. Left QF-3 rings.

Motivated in part by (1.1) we obtain the following characterizations of the injective primitive left ideals in a left QF-3 ring.

(2.1) THEOREM. *Let R be a left* QF-3 *ring. Then the following statements about a primitive left ideal Re of R are equivalent.*

(a) *Re is injective.*

(b) $r(N)e \subseteq l(N)$.

(c) $T(eR)$ *is isomorphic to a minimal right ideal of R.*

Proof. (a) $\Rightarrow$ (b). If the indecomposable module Re is injective then its socle $r(N)e$ must be simple. Suppose $r(N)en \neq 0$ for some $n \in N$. Then $re \to ren$ defines a monomorphism $\rho_n\colon Re \to N$ because $\rho_n \neq 0$ on the only minimal submodule of Re. Hence by injectivity $R = L \oplus \operatorname{Im} \rho_n$ and $\operatorname{Im} \rho_n \subseteq N$, so N contains a nontrivial idempotent. This contradiction proves that $r(N)eN = 0$.

(b) $\Rightarrow$ (c). If $r(N)e \subseteq l(N)$ then $l(N)e \neq 0$ and the semisimple right module $l(N)$ contains a copy of $T(eR)$.

(c) $\Rightarrow$ (a) Suppose $T(eR)$ is isomorphic to a minimal right ideal $S \leqq R_R$. According to (1.1) the sum of the injective primitive left ideals of R is faithful. So $0 \neq SRf = Sf$ for some injective primitive left ideal Rf. But then $T(eR)f \neq 0$ and $Re \cong Rf$.

Note that in proving (2.1) we showed that an injective primitive left ideal Re in any left artinian ring satisfies (b). The converse is not true, even if we demand that $S(Re)$ be simple.

(2.2) Example. Let K be a field and let R be the ring of 4×4 matrices

$$\begin{bmatrix} a & 0 & x & y \\ 0 & b & 0 & z \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & a \end{bmatrix}$$

with entries in K. Then

$$e_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad e_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

are primitive idempotents in R. Observing that $l(N) = N = r(N)$ it is easy to check that $S(Re_2) \cong T(Re_1)$ and that $r(N)e_2 \subseteq l(N)$. However if we let $(\ )^*$ denote the vector space dual it follows that

$$E(Re_2) \cong E(T(Re_1)) \cong (e_1R)^*$$

and

$$c((e_1R)^*) = c(e_1R) > c(Re_2),$$

so Re_2 is not injective.

In view of (1.1) we may restate (2.1) to obtain ideal theoretic methods for determining the minimal faithful module of a left QF-3 ring.

(2.3) Corollary. *If R is a left* QF-3 *ring with basic set of primitive idempotents $e_1, \ldots, e_n$ then the unique minimal faithful left R-module U is isomorphic to the sum of the Re_i, $1 \leqq i \leqq n$, with the property that $r(N)e_i \subseteq l(N)$. Equivalently, U is isomorphic to the sum of those Re_i, $1 \leqq i \leqq n$, such that $T(e_iR)$ is isomorphic to a minimal right ideal of R.*

The final theorem of this section is an analogue of the fact that left and right socles coincide in a QF ring.

(2.4) Lemma. *If R is left* QF-3 *with a primitive left ideal Re that is not injective then $Re \cap l(N) = 0$.*

Proof. Using the notation of (1.1) let $U=Re_1+\cdots+Re_m$ be the unique minimal faithful left module for R. Suppose e is a primitive idempotent in R and Re is not injective. It follows that, for $k=1,\ldots,m$, $eRe_k\subseteq Ne_k$ because Re cannot be isomorphic to the injective module Re_k. Hence $eU\subseteq NU$. Let $r\in R$. If $reN=0$ then $reU=re(eU)\subseteq re(NU)=0$, and so since U is faithful, $Re\cap l(N)=0$.

(2.5) THEOREM. *Let R be left* QF-3. *Then $r(N)\cap l(N)$ is the left ideal generated by the socles of injective left ideals in R. In particular, R is* QF *if and only if $S({}_RR)=S(R_R)$.*

Proof. Let I be the left ideal of R generated by the socles of the injective left ideals of R. Let $J=r(N)\cap l(N)$. If L is an injective left ideal of R, then L must be a direct summand of ${}_RR$, so we may write $L=Rf$, where $f^2=f\in R$. Write $f=f_1+\cdots+f_s$ where the f_i are primitive orthogonal idempotents in R. Then

$$L = Rf = \oplus\sum_{i=1}^{s} Rf_i, \qquad S(L) = \oplus\sum_{i=1}^{s} r(N)f_i$$

and by (2.1), $r(N)f_i\subseteq l(N)$, $i=1,\ldots,s$. This proves that $I\subseteq J$. Let e be a primitive idempotent of R. Then if $Je\neq 0$, we have

$$0 \neq Je = Je\cap l(N) \subseteq Re\cap l(N),$$

so that Re is injective by (2.4). Hence either $Je=0$ or $Je\subseteq r(N)e\subseteq I$ for each primitive idempotent $e\in R$. Thus $J=I$.

The last statement is now obvious, since $r(N)=l(N)$ implies that $J=S({}_RR)$.

3. **Antistrophic primitives and** QF-3 **rings.** Nakayama [10] proved that a ring R with minimum conditions and basic set of primitive idempotents $e_1,\ldots,e_n$ is QF if and only if there is a permutation π of $\{1,\ldots,n\}$ such that for $k=1,\ldots,n$,

(a) Re_k has a simple socle $S(Re_k)\cong T(Re_{\pi(k)})$;

(b) $e_{\pi(k)}R$ has a simple socle $S(e_{\pi(k)}R)\cong T(e_kR)$.

From this it follows that R is QF if and only if for each primitive idempotent $e\in R$ there is a primitive idempotent $f\in R$ such that $S(Re)\cong T(Rf)$ and $S(fR)\cong T(eR)$.

DEFINITION. Let e and f be primitive idempotents in R. If $S(Re)\cong T(Rf)$ and $S(fR)\cong T(eR)$ then we say that Re is *antistrophic* to fR. Moreover, in this case we call Re (fR) a *left* (*right*) *antistrophic primitive* for R.

Note that if e is a primitive idempotent in a finite dimensional algebra R, then $E(T(Re))$ is a vector space dual of eR. Thus one readily shows that the antistrophic primitives in an algebra are precisely its injective primitive one-sided ideals.

The concept of antistrophic primitives allows another characterization of the injective primitive modules for a QF-3 ring.

(3.1) THEOREM. *In a* QF-3 *ring R a primitive left ideal Re is injective if and only if Re is antistrophic to some primitive right ideal.*

Proof. ($\Rightarrow$) Suppose the primitive left ideal Re is injective. Then its socle $r(N)e$ is simple and hence isomorphic to $T(Rf)$ for some primitive idempotent $f \in R$. By the right-hand version of (2.1), fR is injective so $fl(N) = S(fR)$ is simple and by (2.1) itself

$$0 \neq fr(N)e \subseteq fl(N)e.$$

Thus Re is antistrophic to fR.

($\Leftarrow$) If Re is antistrophic to fR then $T(eR) \cong S(fR)$, a minimal right ideal, and by (2.1) Re is injective.

Restating (3.1) in terms of minimal faithful modules we have

(3.2) Corollary. *Let R be a* QF-3 *ring. Let $e_1, \ldots, e_m$ and $f_1, \ldots, f_m$ be sets of mutually orthogonal primitive idempotents such that $Re_1, \ldots, Re_m$ is a complete collection of pairwise nonisomorphic left antistrophic primitives and Re_k is antistrophic to f_kR, $k = 1, \ldots, m$. Let*

$$e = e_1 + \cdots + e_m \quad \text{and} \quad f = f_1 + \cdots + f_m.$$

then Re (fR) is the unique minimal faithful left (*right*) *module for R.*

In addition, (3.1) together with (1.1) serves to establish the following correspondences.

(3.3) Corollary. *If R is a* QF-3 *ring then there are natural* 1-1 *correspondences between the sets of isomorphism classes of*

(a) *indecomposable injective projective left R-modules*;

(b) *minimal left ideals of R*;

(c) *indecomposable injective projective right R-modules*;

(d) *minimal right ideals of R.*

By (1.1) and (3.1) every QF-3 ring has the property that each of its minimal left ideals is isomorphic to the socle of a left antistrophic primitive. We shall say that any two-sided artinian ring with this property has *enough antistrophic primitives.* With the aid of the next two lemmas we are able to prove our main result—QF-3 rings are precisely those rings with enough antistrophic primitives.

(3.4) Lemma. *Let e be a primitive idempotent in R. Then*

$$\mathrm{Ann}_R\,(E(T(Re))) = \mathrm{Ann}_R\,(eR_R).$$

Proof. Let $E = E(T(Re))$ and suppose $I \leqq {}_RR$. Then if $IE \neq 0$, there exists an $x \in E$ such that

$$\rho_x : s \to sx, \qquad s \in I$$

defines a nonzero homomorphism of I into E. But then since $T(Re) = S(E)$ is simple and essential in E, Im ρ_x contains a copy of $T(Re)$. Hence $IE \neq 0$ implies I has a composition factor isomorphic to $T(Re)$. Conversely, suppose $J < L \leqq I$ and let

$\eta: L \to L/J$ be the natural epimorphism. If $\phi: L/J \to T(Re) \leqq E$ is an isomorphism then $\phi \circ \eta$ is a nonzero homomorphism from a left ideal of R into E. Thus, by injectivity, $IE \geqq LE \geqq Lx = \phi \circ \eta(L) \neq 0$, for some $x \in E$.

Now $\mathrm{Ann}_R(E)E = 0$, so as a left module, $\mathrm{Ann}_R(E)$ has no composition factors isomorphic to $T(Re)$. Hence

$$eR(\mathrm{Ann}_R(E)) = e(\mathrm{Ann}_R(E)) = 0.$$

On the other hand, $e\,\mathrm{Ann}_R(eR) = 0$ so the left ideal $\mathrm{Ann}_R(eR)$ has no composition factors isomorphic to $T(Re)$. Therefore $(\mathrm{Ann}_R(eR))E = 0$. This proves the lemma.

Observe that (3.4) is valid for R any semiperfect ring (i.e., R/N is semisimple and idempotents along with orthogonality relations can be lifted modulo N [1]). Also from (3.4) it follows that if e is a primitive idempotent in R then $E(T(Re))$ has a composition factor isomorphic to $T(Rf)$ if and only if $T(fR)$ is isomorphic to a composition factor of eR; and the number of terms in the Loewy series for $E(T(Re))$ and eR are equal.

(3.5) Lemma. *Let R have enough antistrophic primitives. Let $e = e_1 + \cdots + e_m$ and $f = f_1 + \cdots + f_m$ be as in the hypothesis of* (3.2). *Then*

(a) *Re (fR) is a faithful left (right) R-module;*

(b) *$fr(N)e = fl(N)e$.*

Proof. (a) Since Re_k is antistrophic to f_kR, $1 \leqq k \leqq m$, we have, by (3.4) and its right-left dual version,

$$\begin{aligned}\mathrm{Ann}_R(f_kR) &= \mathrm{Ann}_R(E(T(Rf_k))) = \mathrm{Ann}_R(E(Re_k))\\ &\leqq \mathrm{Ann}_R(Re_k) = \mathrm{Ann}_R(E(T(e_kR)))\\ &= \mathrm{Ann}_R(E(f_kR)) \leqq \mathrm{Ann}_R(f_kR).\end{aligned}$$

So that for $k = 1, \ldots, m$,

$$\mathrm{Ann}_R(E(Re_k)) = \mathrm{Ann}_R(Re_k) = \mathrm{Ann}_R(f_kR).$$

Now, noting that $E(Re_1) \oplus \cdots \oplus E(Re_m)$ is an injective left R-module containing copies of all the minimal left ideals of R, we have

$$0 = \bigcap_{k=1}^{m} \mathrm{Ann}_R(E(Re_k)) = \bigcap_{k=1}^{m} \mathrm{Ann}_R(Re_k) = \bigcap_{k=1}^{m} \mathrm{Ann}_R(f_kR).$$

This proves that Re and fR are faithful.

(b) Since $k, j \in \{1, \ldots, m\}$, $k \neq j$, implies that

$$f_kr(N)e_j = 0 = f_kl(N)e_j,$$

it is sufficient to show that

$$f_kr(N)e_k = f_kl(N)e_k, \qquad k = 1, \ldots, n.$$

Suppose not—say $f_kr(N)e_kN \neq 0$. Then, since Re_k is antistrophic to f_kR, the right ideal $f_k r(N)e_k N$ must contain a copy of $T(e_k R)$. That is, $r(N)e_k N e_k \neq 0$. So for some $n \in N$ right multiplication by ne_k induces a monomorphism

$$0 \to Re_k \to Ne_k.$$

This contradicts the fact that $c(Re_k)$ is finite. By symmetry, the proof is complete.

Suppose R is a ring with enough antistrophic primitives. Let e and f be as in (3.5). Then since fR is faithful we see that "enough antistrophic primitives" is automatically a two-sided condition in the sense that each minimal right ideal of R is isomorphic to the socle of a right antistrophic primitive. Moreover Re (fR) is a minimal faithful left (right) R-module because none of its proper direct summands contains copies of every minimal left (right) ideal. Thus the problem of proving that such a ring is QF-3 is reduced to showing that Re (fR) is the *unique* minimal faithful left (right) R-module or equivalently we must show that Re is injective.

(3.6) THEOREM. *For a ring R with both minimum conditions, the following are equivalent.*

(a) *R is* QF-3.

(b) *R has enough antistrophic primitives.*

(c) *There exist idempotents e and f in R such that*

(i) *Re and fR are faithful R-modules;*

(ii) *the functors*

$$\mathrm{Hom}_{fRf}(\ \ , fRe) \quad \textit{and} \quad \mathrm{Hom}_{eRe}(\ \ , fRe)$$

define a duality between the category of finitely generated left fRf-modules and the category of finitely generated right eRe-modules.

Proof. (a) $\Rightarrow$ (b). This implication is an immediate consequence of (1.1) and (3.1).

(b) $\Rightarrow$ (c). Let R, e and f be as in (3.5). Then Re and fR are faithful. A standard argument shows that fRf and eRe both have minimum conditions, and that fRe is finitely generated and faithful both as a left fRf-module and as a right eRe-module. Thus according to [8, Theorem 6.3] we need only show that, for each simple left fRf (right eRe)-module T, Hom (T, fRe) is simple as a right eRe (left fRf)-module. According to [6, p. 48, Proposition 1] fNf is the radical of fRf. Hence $T(fRf_k) = fRf_k/fNf_k$, $k=1,\dots,m$, is a typical simple left fRf-module and $S({}_{fRf}fRe) = \mathrm{Ann}_{fRe}(fNf)$. Let $S = fSe = S({}_{fRf}fRe)$. Then $fNf \cdot fr(N)e \subseteq Nr(N) = 0$ so $fr(N)e \subseteq S$. On the other hand, suppose $S \nsubseteq r(N)$. Then for some $k \in \{1,\dots,m\}$, $0 \neq NSe_k = (1-f)NSe_k + fNSe_k = (1-f)NSe_k$. But, since $r(N)e_k$ is the unique minimal R-submodule of Re_k, this implies that $fr(N)e_k = 0$, contrary to the definition of e and f. Thus we have shown that $S({}_{fRf}fRe) = fr(N)e$. Now for each $a \in f_k r(N)e$ define $\rho_a : T(fRf_k) \to fr(N)e$ via

$$\rho_a(s + fNf_k) = sa, \qquad s \in fRf_k.$$

Then $a \to \rho_a$ is an eRe-isomorphism $f_k r(N)e \to \operatorname{Hom}_{fRf}(T(fRf_k), fr(N)e)$. Thus

$$\begin{aligned}\operatorname{Hom}_{fRf}(T(fRf_k), fRe) &= \operatorname{Hom}_{fRf}(T(fRf_k), S(fRe))\\ &= \operatorname{Hom}_{fRf}(T(fRf_k), fr(N)e) \cong f_k r(N)e\\ &= f_k l(N)e\end{aligned}$$

as right eRe-modules (the last equality follows from (3.5), (b)). Moreover, since e does not annihilate the simple right R-module $f_k l(N)$, one can easily check that $f_k l(N)e$ is simple over eRe. A symmetric argument now completes this part of the proof.

(c) $\Rightarrow$ (a). Suppose Re and fR are faithful and $(\ \ )^* = \operatorname{Hom}(\ \ , fRe)$ defines a duality between the category of finitely generated left fRf-modules and the category of finitely generated right eRe-modules. We shall show that the faithful projective left R-module Re is injective. According to [8, Theorem 6.3] our hypotheses imply that fRe_{eRe} is injective and that each simple right eRe-module is isomorphic to an eRe-submodule of fRe. Thus

$$Re_{eRe} = fRe \oplus (1-f)Re$$

satisfies the hypothesis of [8, Theorem 16.4]. So, considering $\hat{R} = \operatorname{Hom}_{eRe}(Re, Re)$ as a ring of left operators on Re, Re is a faithful injective left $\hat{R}$-module. Now note that $\lambda\colon R \to \hat{R}$ via

$$[\lambda(r)](se) = rse, \qquad r \in R,\ se \in Re,$$

is a unital ring monomorphism and that if $\hat{r} \in \hat{R}$ then

$$\begin{aligned}[\hat{r}\lambda(re)](se) &= \hat{r}(rese) = \hat{r}(re)ese\\ &= [\lambda(\hat{r}re)](se)\end{aligned}$$

for all $re, se \in Re$. Hence $\lambda|_{Re}$ is a left $\hat{R}$ monomorphism, and we have

$$\lambda(Re) \subseteq \hat{R}\lambda(e) \subseteq \hat{R}\lambda(Re) \subseteq \lambda(Re).$$

Thus we view R as a unital subring of $\hat{R}$ with $Re = \hat{R}e$. If $\hat{r} \in \hat{R}$ then $f\hat{r}\colon Re \to fRe$, so as left fRf modules

$$f\hat{R} \leqq \operatorname{Hom}_{eRe}(Re, fRe) = (Re)^*.$$

And, on the other hand,

$$\rho\colon Re \to \operatorname{Hom}_{fRf}(fR, fRe) = (fR)^*$$

defined via

$$[\rho(re)](fs) = fsre, \qquad fs \in fR,\ re \in Re$$

is a right eRe monomorphism. Now, since a duality must preserve composition lengths, we have

$$c({}_{fRf}f\hat{R}) \leqq c({}_{fRf}(Re)^*) = c(Re_{eRe}) \leqq c((fR)^*_{eRe}) = c({}_{fRf}fR).$$

This, along with the fact that $fR \leqq f\hat{R}$ as left fRf modules, implies that $fR = f\hat{R}$. Now to show that ${}_RRe$ is injective, let I be a left ideal of R and suppose that $g: I \to Re$ is an R-homomorphism. If $\hat{r}_i \in \hat{R}$ and $a_i \in I$, $i = 1, \dots, n$, let

$$\bar{g}\left(\sum \hat{r}_i a_i\right) = \sum \hat{r}_i g(a_i).$$

Suppose $\sum \hat{r}_i a_i = 0$. Then for all $ft \in fR$ we have, since $ft\hat{r}_i \in f\hat{R} = fR \subseteq R$,

$$0 = g(0) = g\left(ft \sum \hat{r}_i a_i\right) = \sum ft\, \hat{r}_i g(a_i) = ft\left(\sum \hat{r}_i g(a_i)\right).$$

Thus, recalling that fR_R is faithful, we see that $\bar{g}$ is a well defined $\hat{R}$ homomorphism $\bar{g}: \hat{R}I \to Re$. Since ${}_{\hat{R}}Re$ is injective there exists an $x \in Re$ such that $\bar{g}(\hat{a}) = \hat{a}x$ for all $\hat{a} \in \hat{R}I$. In particular, $g(a) = \bar{g}(a) = ax$, for all $a \in I$. This shows that ${}_RRe$ is injective and the theorem is proved.

Making one more application of Morita's duality theorem [8, Theorem 6.3] we have a corollary which is the QF-3 analogue of the annihilator characterization of QF rings.

(3.7) Corollary. *A ring R with minimum conditions is* QF-3 *if and only if there exist idempotents e and f in R that satisfy*

(a) *Re and fR are faithful R-modules*;

(b) *for every left ideal $I \leqq fRf$ and for every right eRe-submodule $W \leqq fRe$*

$$I = \mathrm{Ann}_{fRf}(\mathrm{Ann}_{fRe}(I)) \quad \textit{and} \quad W = \mathrm{Ann}_{fRe}(\mathrm{Ann}_{fRf}(W));$$

(c) *for every right ideal $J \leqq eRe$ and for every left fRf-submodule $V \leqq fRe$*

$$J = \mathrm{Ann}_{eRe}(\mathrm{Ann}_{fRe}(J)) \quad \textit{and} \quad V = \mathrm{Ann}_{fRe}(\mathrm{Ann}_{eRe}(V)).$$

Note that part (c) of (3.6) and the conditions of (3.7) may be viewed as duality relationships between the left and the right minimal faithful modules over a QF-3 ring. Observing that the maps ρ and $\lambda|_{fR}$ of the proof of (c) $\Rightarrow$ (a) of (3.6) are in fact R-isomorphisms, we see that these modules are dual to one another in the following sense.

(3.8) Proposition. *Let R be a* QF-3 *ring with minimal faithful left* (*right*) *module Re* (*fR*). *Then as R-modules,*

$$Re \cong \mathrm{Hom}_{fRf}(fR, fRe) \quad \textit{and} \quad fR \cong \mathrm{Hom}_{eRe}(Re, fRe).$$

4. QF-2 **rings.** Thrall [12] showed that in a QF-2 algebra every primitive left (right) ideal is either injective or can be imbedded in an injective primitive left (right) ideal. Using this he showed that every QF-2 algebra is QF-3. In this section we prove that, in fact, any QF-2 ring is QF-3 by showing that QF-2 rings have enough antistrophic primitives.

(4.1) Theorem. *Every* QF-2 *ring is* QF-3.

Proof. Let R be a QF-2 ring with basic set of primitive idempotents $e_1, \ldots, e_n$. Let S be a minimal left ideal of R. Since R is QF-2 there is an $i \in \{1, \ldots, n\}$ such that S is isomorphic to the simple socle $r(N)e_i$ of Re_i. Choose $k \in \{1, \ldots, n\}$ such that $c(Re_k)$ is maximal with respect to $S \cong S(Re_k)$. Suppose $r(N)e_kN \neq 0$. Then, since $Re_1 + \cdots + Re_n$ is faithful, $r(N)e_kNe_j \neq 0$ for some $j = 1, \ldots, n$. So right multiplication by some element of Ne_j gives a monomorphism of Re_k into $Ne_j < Re_j$. But then $c(Re_j) > c(Re_k)$ and, since R is QF-2, $S = S(Re_j)$. This contradiction of our choice of k proves that $r(N)e_k \subseteq l(N)$. Now let $t \in \{1, \ldots, n\}$ with $T(Re_t) \cong S$. Then $0 \neq e_t r(N)e_k \subseteq e_t l(N)e_k$, so that the simple socle $e_t l(N)$ of e_tR is isomorphic to $T(e_kR)$. Thus Re_k is antistrophic to e_tR and the theorem is proved.

The proof of the following generalization of Thrall's Theorem 1 on QF-2 algebras [12, §3] is an immediate consequence of (4.1) and elementary properties of injective modules over artinian rings.

(4.2) Corollary. *A ring with minimum conditions is* QF-2 *if and only if each of its primitive one-sided ideals is either injective or isomorphic to a submodule of an injective primitive one-sided ideal.*

As we noted earlier, Harada [5] has shown that for rings with both minimum conditions left QF-3 is equivalent to right QF-3. Consider the algebra R of 3×3 matrices of the form

$$\begin{bmatrix} a & x & y \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$$

with entries in some field K. Every primitive left ideal of R has a simple socle, but the primitive right ideal generated by

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

does not. Note that R does not have enough antistrophic primitives and so is not QF-3. (This fact also follows from an argument given in [14].) These observations lead to a question: Are QF-3 rings in which every primitive left ideal has a simple socle QF-2? We know of no examples to the contrary.

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